SURFACE AREA. I

BY

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1. **Introduction.** We define a measure Φ over E_3, which can be thought of as a generalization of Carathéodory linear measure, and prove that $\Phi(S)$ equals the Lebesgue area of the surface defined by f above R, whenever f is a continuous numerically valued function on E_2, R is a rectangle and

$$S = E_3 \underset{x}{E} [(x_1, x_2) \in R, x_3 = f(x_1, x_2)].$$

We feel that Theorem 4.1 is of intrinsic interest. It has been proved by Banach in several special cases. We shall use it again in *Surface area.* II where we discuss a certain integral identity concerning parametrically given m-dimentional surfaces in n-space.

2. **Definitions.**

2.1. *Definition.* If f and g are functions, then $g:f$ is the function h such that $h(x) = g[f(x)]$ for every x.

If f is a function, then

$$f^*(S) = \underset{y}{E} [y = f(x) \text{ for some } x \in S].$$

If S is a set, then $p(S)$ = the number of elements in S whenever S is finite; otherwise $p(S) = \infty$.

If f is a function, S is a set, and y is a point, then

$$N(f, S, y) = p\{S \cdot \underset{x}{E} [f(x) = y]\}.$$

It follows immediately that

$$N[f, g^*(S), y] = N(f:g, S, y)$$

whenever f is a function and g is a univalent function.

2.2. *Definition.* If S_x is a set for each $x \in F$, then

$$\sum_{x \in F} S_x = \underset{y}{E} [y \in S_x \text{ for some } x \in F].$$

If F is a family of sets, then

$$\sigma(F) = \sum_{x \in F} x.$$

If $a_x \geqq 0$ for each x in a countable set C, then

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$$\sum_{x \in C} a_x$$

denotes the obvious numerical sum, finite or infinite.

It will in each case be clear from context whether point set or numerical summation is intended.

2.3. *Definition.* We say ϕ is a *measure* over B if and only if ϕ is a function whose domain is the set of all subsets of B and which satisfies the conditions:

(i) $0 \leqq \phi(S) \leqq \infty$ for $S \subset B$,

(ii) $\phi(0) = 0$,

(iii) $\phi(S) \leqq \phi(T)$ whenever $S \subset T \subset B$,

(iv) $\phi[\sigma(F)] \leqq \sum_{S \in F} \phi(S)$ for every countable family F of subsets of B.

Following Carathéodory, we say a set S is ϕ *measurable* if and only if $S \subset B$ and

$$\phi(T) = \phi(TS) + \phi(T - S) \text{ whenever } T \subset B.$$

If f is a ϕ measurable function and S is a ϕ measurable set, then the (Lebesgue) integral of f relative to ϕ over the set S is denoted by

$$\int_S f(x) d\phi x.$$

We also abbreviate

$$\int_B f(x) d\phi x = \int f(x) d\phi x.$$

2.4. *Definition.* G is a *partition* of A if and only if G is a countable disjointed family with $\sigma(G) = A$.

If f is a function on A to B, ϕ is a measure over B, and F is a family of subsets of $T \subset A$, then

$$V_F(f, T, \phi)$$

is the supremum of numbers of the form

$$\sum_{S \in G} \phi[f^*(S)]$$

where $G \subset F$ and G is a partition of T.

2.5. *Definition.* If S is a subset of a metric space, then we denote its diameter by diam S.

We say F *covers* A *in the sense of Vitali* if and only if A is a metric space and F is such a family of subsets of A that corresponding to each $x \in A$ and each $\epsilon > 0$ there is a set S for which $x \in S \in F$ and diam $S < \epsilon$.

2.6. *Definition.* Euclidean n-space is denoted by E_n and we write $x = (x_1, x_2, \cdots, x_n)$ for $x \in E_n$.

If $S\subset E_n$, then $|S|$ denotes the n-dimensional Lebesgue measure of S. We further abbreviate

$$\int_S f(x)d\phi x = \int_S f(x)dx, \qquad \int f(x)d\phi x = \int f(x)dx$$

whenever ϕ is n-dimensional Lebesgue measure.

2.7 *Definition.* If $a<b$, then

$$[a, b] = \underset{x}{E}\,[a \leqq x \leqq b], \qquad [a, b]^0 = \underset{x}{E}\,[a < x < b].$$

If $a<b$ and f is a function whose domain includes the closed interval $[a, b]$, then $T_a^b f$ is the supremum of numbers of the form

$$\sum_{i=1}^{n} | f(t_i) - f(t_{i-1}) |$$

where $a=t_0\leqq t_1\leqq t_2\leqq \cdots \leqq t_n=b$.

If $a\geqq b$, then $T_a^b f=0$. We also write

$$T_a^b f = T_{t=a}^b f(t).$$

If f is a function on E_n to E_1 and j is a positive integer not greater than n, then $D_j f$ is the function such that

$$D_j f(x) = \lim_{h\to 0} \frac{f(x_1, \cdots, x_{j-1}, x_j + h, x_{j+1}, \cdots, x_n) - f(x)}{h}.$$

3. The measure Φ.

3.1 *Definition.* For each $c\in E_3$ with $|c|=1$ we define the *projecting function* P_c as follows:

If $c_1^2+c_2^2\geqq 0$, then

$$P_c(y) = (c_1^2 + c_2^2)^{-1/2}(c_2y_1 - c_1y_2, c_3(c_1y_1 + c_2y_2) - (c_1^2 + c_2^2)y_3)$$

for each $y\in E_3$.

If $c_1=c_2=0$, then

$$P_c(y) = (y_1, y_2) \quad \text{for} \quad y \in E_3.$$

Evidently P_c has domain E_3 and range E_2.

We might say, geometrically, that P_c effects a perpendicular projection onto a plane whose normal has direction c. Unfortunately we shall have to use the explicit formula.

3.2 *Definition.* For $S\subset E_3$ we define

$$\gamma(S) = \sup_{c\in E_3, |c|=1} | P_c^*(S) |.$$

We see that $\gamma(S)$ is the supremum of the areas of the perpendicular projections of S onto planes in all directions.

3.3 *Definition.* Suppose $A\subset E_3$.

For each $r>0$ we define $\gamma_r(A)$ to be the infimum of numbers of the form

$$\sum_{S\in F}\gamma(S)$$

where F is such a countable family of open connected subsets of E_3, each of diameter less than r, that $A\subset\sigma(F)$.

We note that $0<r_2<r_1$ implies $\gamma_{r_1}(A)\leqq\gamma_{r_2}(A)$ and define(1)

$$\Phi(A) = \lim_{r\to 0+}\gamma_r(A).$$

3.4 THEOREM. (i) *Φ is a measure over E_3.*

(ii) *If $A\subset E_3$, $B\subset E_3$ and A and B are a positive distance apart, then* $\Phi(A+B)=\Phi(A)+\Phi(B)$.

(iii) *Closed subsets of E_3 are Φ-measurable.*

We omit the proof, which is easily given by well known methods.

3.5 THEOREM. *If A is Φ measurable, $\Phi(A)<\infty$ and $\epsilon>0$, then there is a closed set C such that*

$$C\subset A \quad \textit{and} \quad \Phi(A-C)<\epsilon.$$

We refer to A. P. Morse and J. F. Randolph, *The ϕ rectifiable subsets of the plane*, Theorem 3.13 and the remark following 3.4(2), for the general theorem of which this is a special case.

In this paper we make no use of Theorem 3.5, but we believe that it will be of great help in further investigations of Φ.

3.6 THEOREM. *If $A\subset E_3$ and $c\in E_3$, $|c|=1$, then*

$$|P_c^*(A)|\leqq\Phi(A).$$

Proof. If F is any countable family for which $A\subset\sigma(F)$, then

$$|P_c^*(A)|\leqq|P_c^*[\sigma(F)]|=\left|\sum_{S\in F}P_c^*(S)\right|\leqq\sum_{S\in F}|P_c^*(S)|\leqq\sum_{S\in F}\gamma(S).$$

Reference to 3.3 completes the proof.

4. Generalization of a theorem of Banach.

4.1 THEOREM. *If f is a function with domain A and range B, ϕ is a measure*

(1) Of all the other surface measures which have been considered in the literature, the one defined by Carathéodory (*Über das lineare Muss von Punktmengen—eine Verallgemeinerung des Längenbegriffs*, Nachr. Ges. Wiss. Göttingen, 1914) is most similar to our measure Φ.

(2) Trans. Amer. Math. Soc. vol. 55 (1944) pp. 236–305.

over B, F covers A in the sense of Vitali, $S \in F$ implies $f^*(S)$ is ϕ measurable, there is a sequence G of partitions of A such that

$$G_n \subset F \quad \text{for} \quad n = 1, 2, 3, \cdots,$$

$$\left\{ \sup_{S \in G_n} (\operatorname{diam} S) \right\} \to 0 \quad \text{as} \quad n \to \infty;$$

then

$$\int N(f, A, y) d\phi y = V_F(f, A, \phi) = \lim_{n \to \infty} \left\{ \sum_{S \in G_n} \phi[f^*(S)] \right\}.$$

Proof. Setting

$$K(T, y) = 1 \quad \text{if} \quad y \in T, \qquad K(T, y) = 0 \quad \text{if} \quad y \notin T,$$

we divide the proof into four parts.

Part 1. If $y \in B$, then $N(f, A, y) = \lim_{n \to \infty} \{\sum_{S \in G_n} K[f^*(S), y]\}$.

Let m be any integer for which

$$m \leqq N(f, A, y).$$

Then there is a set α with $p(\alpha) = m$ and

$$f(x) = y \quad \text{for} \quad x \in \alpha.$$

Choose ν so large that, for $n > \nu$, each member of G_n has diameter less than the smallest distance between any two distinct points of α; hence m distinct members of G_n contain points of α and the image of each of these sets contains y. We conclude

$$m \leqq p\{G_n \mathop{E}_{S} [y \in f^*(S)]\} = \sum_{S \in G_n} K[f^*(S), y] \quad \text{for} \quad n > \nu,$$

and consequently

$$m \leqq \liminf_{n \to \infty} \left\{ \sum_{S \in G_n} K[f^*(S), y] \right\}.$$

From the arbitrary nature of m we now infer that

$$N(f, A, y) \leqq \liminf_{n \to \infty} \left\{ \sum_{S \in G_n} K[f^*(S), y] \right\}.$$

On the other hand we have obviously

$$\sum_{S \in G_n} K[f^*(S), y] \leqq N(f, A, y)$$

for each positive integer n. Part 1 is now evident.

Part 2. $\int N(f, A, y) d\phi y \leqq \liminf_{n \to \infty} \{\sum_{S \in G_n} \phi[f^*(S)]\}$.

From Part 1 and Fatou's lemma we infer:

$$\liminf_{n\to\infty}\left\{\sum_{S\in G_n}\phi[f^*(S)]\right\} = \liminf_{n\to\infty}\left\{\sum_{S\in G_n}\int K[f^*(S), y]d\phi y\right\}$$
$$= \liminf_{n\to\infty}\left\{\int\sum_{S\in G_n}K[f^*(S), y]d\phi y\right\}$$
$$\geqq \int\left\{\liminf_{n\to\infty}\sum_{S\in G_n}K[f^*(S), y]\right\}d\phi y$$
$$= \int N(f, A, y)d\phi y.$$

Part 3. $V_F(f, A, \phi) \leqq \int N(f, A, y)d\phi y$.

Let H be any partition of A for which $H\subset F$. Then

$$\sum_{S\in H}\phi[f^*(S)] = \sum_{S\in H}\int K[f^*(S), y]d\phi y$$
$$= \int\sum_{S\in H}K[f^*(S), y]d\phi y \leqq \int N(f, A, y)d\phi y.$$

The arbitrary nature of H completes the proof.

Part 4. $\int N(f, A, y)d\phi y = V_F(f, A, \phi) = \lim_{n\to\infty}\{\sum_{S\in G_n}\phi[f^*(S)]\}$.

We use Part 2, the conditions satisfied by G, and Part 3 to conclude:

$$\int N(f, A, y)d\phi y \leqq \liminf_{n\to\infty}\left\{\sum_{S\in G_n}\phi[f^*(S)]\right\}$$
$$\leqq \limsup_{n\to\infty}\left\{\sum_{S\in G_n}\phi[f^*(S)]\right\} \leqq V_F(f, A, \phi)$$
$$\leqq \int N(f, A, y)d\phi y.$$

4.2 Theorem. *If f is a numerically valued continuous function on the closed interval $[a, b]$, then*

$$\int N\{f, [a, b], y\}dy = T_a^b f.$$

Proof. Let F be the family of all subintervals (non-empty connected subsets) of $[a, b]=A$. Let ϕ be one-dimensional Lebesgue measure. Let G be such a sequence of finite partitions of A that

$$G_n \subset F \quad \text{for} \quad n = 1, 2, 3, \cdots,$$
$$\left\{\sup_{S\in G_n}(\operatorname{diam} S)\right\}\to 0 \quad \text{as} \quad n\to\infty.$$

The preceding theorem implies

$$\lim_{n\to\infty}\left\{\sum_{S\in G_n}|f^*(S)|\right\}=\int N(f,A,y)dy.$$

On the other hand the continuity of f insures

$$\begin{aligned} T_a^b f &= \lim_{n\to\infty}\left\{\sum_{S\in G_n}|f(\sup S)-f(\inf S)|\right\} \\ &\leqq \limsup_{n\to\infty}\left\{\sum_{S\in G_n}|f^*(S)|\right\} = \limsup_{n\to\infty}\left\{\sum_{S\in G_n}|\sup f^*(S)-\inf f^*(S)|\right\} \\ &\leqq T_a^b f. \end{aligned}$$

4.3 *Remark.* Theorem 4.2 is due to Banach(3) and may be generalized as follows:

Suppose B is a metric space (for instance E_2) and ϕ is Carathéodory linear measure(4) *over B. If f is continuous on $[a, b]$ to B, then*

$$\int N\{f,[a,b],y\}d\phi y=T_a^b f.$$

We outline a simple proof:
Define F and G as in the proof of 4.2, use the well known inequality

$$|f(\beta)-f(\alpha)|\leqq\phi\{f^*([\alpha,\beta])\}\leqq T_\alpha^\beta f\quad\text{for}\quad a\leqq\alpha\leqq\beta\leqq b$$

and apply 4.1.

4.4 THEOREM. *If $A\subset E_3$, A is of class F_σ (or, more generally, A is an analytic set*(5)) *and $a\in E_3$ with $|a|=1$, then*

$$\Phi(A)\geqq\int N(P_a,A,y)dy.$$

Proof. Let F be the family of all those subsets of A which are bounded and of class F_σ (which are analytic). Let ϕ be plane Lebesgue measure and let g be a function such that $g(x)=x$ for $x\in A$.

Then $S\in F$ implies $P_a^*(S)$ is of class F_σ (is analytic) and hence ϕ measurable. Also $\Phi[g^*(S)]=\Phi(S)\geqq\phi[P_a^*(S)]$ in virtue of 3.6.

(3) S. Banach, *Sur les lignes rectifiables et les surfaces dont l'aire est finie*, Fund. Math. vol. 7 (1925).

(4) Loc. cit. footnote 1.

(5) In this paper we use only the special case in which A is of class F_σ. The properties of analytic sets which are used in the proof of the more general theorem can be found in Hausdorff's *Mengenlehre* 3rd edition, Berlin and Leipzig, 1935, and in Saks's *Theory of the Integral*, Warsaw, 1937.

From this and 4.1 we conclude

$$\Phi(A) = \int N(g, A, y)d\Phi y = V_F(g, A, \Phi) \geqq V_F(P_a, A, \phi)$$

$$= \int N(P_a, A, y)dy.$$

5. Auxiliary interval functions.

5.1 *Notation.* If $a_1 < b_1$, $a_2 < b_2$,

$$R = E_2 \underset{x}{E} [a_1 < x_1 < b_1, a_2 < x_2 < b_2],$$

and if f is continuous on E_2 to E_1, then we write

$$G_1(f, R) = \int_{a_1}^{b_1} | f(u, b_2) - f(u, a_2) | du,$$

$$G_2(f, R) = \int_{a_2}^{b_2} | f(b_1, v) - f(a_1, v) | dv,$$

$$G(f, R) = \{ [G_1(f, R)]^2 + [G_2(f, R)]^2 + | R |^2 \}^{1/2}.$$

This notation agrees with Saks([6]), page 171.

5.2 *Definition.* If S is a convex subset of E_2 and f is continuous on E_2 to E_1, then we denote by

$$H(f, S)$$

the supremum of numbers of the form

$$\sum_{R \in F} G(f, R)$$

where F is a finite disjointed family of open rectangles contained in S.

Analogously H_1 and H_2 are defined in terms of G_1 and G_2.

5.3 *Remark.* If R is a closed rectangle and f is continuous on E_2 to E_1, then the numbers

$$H(f, R), \qquad H_1(f, R), \qquad H_2(f, R)$$

are equal to the numbers denoted by the same symbols in Saks, page 174. Hence our functions H, H_1, H_2 are extensions of the corresponding functions of Saks.

This follows immediately from the definitions and Minkowski's inequality. We omit the proof.

5.4 LEMMA. *If f is continuous on E_2 to E_1, S is a convex subset of E_2 and α*

([6]) S. Saks, *Theory of the integral*, Warsaw, 1937. We shall frequently refer to chap. 5 of this book.

and β are functions such that

$$S = E_2 \mathop{E}_{x} [\alpha(x_1) < x_2 < \beta(x_1)],$$

then

$$H_1(f, S) = \int T^{\beta(u)}_{v=\alpha(u)} f(u, v) du.$$

The argument given by Saks at the bottom of page 174 can readily be transformed into a proof of this lemma.

5.5 Lemma. *If A is a convex subset of E_2 and g is continuous on E_2 to E_1, then*

$$\{[H_1(g, A)]^2 + [H_2(g, A)]^2 + |A|^2\}^{1/2} \leqq H(g, A).$$

Proof. We assume $H(g, A) < \infty$.

Clearly this implies

$$H_1(g, A) < \infty, \qquad H_2(g, A) < \infty, \qquad |A| < \infty.$$

Let $\epsilon > 0$.

Select such a finite disjointed family F of open rectangles contained in A that

$$H_1(g, A) < \sum_{R \in F} G_1(f, R) + \epsilon,$$

$$H_2(g, A) < \sum_{R \in F} G_2(f, R) + \epsilon,$$

$$|A| < \sum_{R \in F} |R| + \epsilon.$$

This can be done by the usual method of combining three subdivisions. Thus

$$\begin{aligned} &\{H_1(g, A)^2 + H_2(g, A)^2 + |A|^2\}^{1/2} \\ &\leqq \left\{\left[\sum_{R \in F} G_1(g, R) + \epsilon\right]^2 + \left[\sum_{R \in F} G_2(g, R) + \epsilon\right]^2 + \left[\sum_{R \in F} |R| + \epsilon\right]^2\right\}^{1/2} \\ &\leqq \sum_{R \in F} \{G_1(g, R)^2 + G_2(g, R)^2 + |R|^2\}^{1/2} + 3^{1/2}\epsilon \\ &= \sum_{R \in F} G(g, R) + \epsilon 3^{1/2} \leqq H(g, A) + \epsilon 3^{1/2}. \end{aligned}$$

Since ϵ was an arbitrary positive number, the proof is complete.

5.6 *Remark.* We define the Lebesgue area of a continuous surface on a convex plane set in the spirit of Saks, page 164, with the obvious modifications.

It is clear that Radó's theorem (see Saks, page 179) still holds with a rectangle replaced by an arbitrary bounded convex set. In fact the proof requires only trivial changes.

Now obviously Lebesgue surface area is invariant under rotation. From the theorem just mentioned it hence follows that

$$H(f:T, B) = H[f, T^*(B)]$$

whenever f is continuous on E_2 to E_1, T is a rotation of the plane and B is a bounded convex plane set.

6. The equality of our measure and Lebesgue area for non-parametric continuous surfaces.

6.1 *Definition.* If n is a positive integer and f is a continuous numerically valued function on E_2, then K_n, f_n, $\bar{f}$ are defined by the relations

$$K_n = E_2 \underset{z}{E} [|z| < n^{-1}],$$

$$f_n(x) = n^2\pi^{-1} \int_{K_n} f(x+z)dz,$$

$$\bar{f}(x) = (x_1, x_2, f(x)).$$

Clearly $\bar{f}$ is a continuous function on E_2 to E_3.

For $A \subset E_2$ and $z \in E_2$ we write

$$A_z = \underset{x}{E} [x - z \in A]$$

and note that A_z is the translate of A by the vector z.

6.2 Lemma. *If B is a bounded open convex subset of E_2, α and β are such functions that*

$$B = E_2 \underset{x}{E} [\alpha(x_1) < x_2 < \beta(x_1)];$$

g is continuous on E_2 to E_1; $\delta > 0$,

$$S = E_3 \underset{z}{E} [(z_1, z_2) \in B, \ |z_3 - g(z_1, z_2)| < \delta],$$
$$a \in E_3, \quad |a| = 1, \quad a_1 = 0, \quad a_2 \geqq 0, \quad a_3 \geqq 0;$$

then $|P_a^*(S)| \leqq H(g, B) + 2\delta$ diam B, $\Phi[\bar{g}^*(B)] \geqq \int T_{v=\alpha(u)}^{\beta(u)} [a_2 g(u, v) - a_3 v]du$.

Proof. Since

(1) $$P_a(y) = (y_1, a_3y_2 - a_2y_3) \quad \text{for} \quad y \in E_3$$

we have

(2) $$[P_a:\bar{g}](x) = (x_1, a_3x_2 - a_2g(x)) \quad \text{for} \quad x \in E_2.$$

For each number u let h_u be the function on $[\alpha(u), \beta(u)]$ such that

$$h_u(v) = a_3v - a_2g(u, v) \quad \text{for} \quad \alpha(u) \leqq v \leqq \beta(u).$$

For $T \subset E_2$ we denote

$$T^u = \underset{v}{E} [(u, v) \in T].$$

The lemma is a consequence of the third and the last of the six parts into which we divide the remainder of the argument.

Part 1. If $(u, v) \in E_2$, then $N[P_a, \bar{g}^*(B), (u, v)] = N\{h_u, [\alpha(u), \beta(u)]^0, v\}$.

$$
\begin{aligned}
N[P_a, \bar{g}^*(B), (u, v)] &= N[P_a:\bar{g}, B, (u, v)] \\
&= p\{B \cdot \underset{(s,t)}{E}[[P_a:\bar{g}](s, t) = (u, v)]\} \\
&= p\{\underset{(s,t)}{E}[\alpha(s) < t < \beta(s), s = u, h_s(t) = v]\} \\
&= p\{\underset{t}{E}[\alpha(u) < t < \beta(u), h_u(t) = v]\} \\
&= N\{h_u, [\alpha(u), \beta(u)]^0, v\}.
\end{aligned}
$$

Part 2. $\int N[P_a, \bar{g}^*(B), y]dy = \int T_{v=\alpha(u)}^{\beta(u)}[a_2 g(u, v) - a_3 v]du$.

With the help of Part 1 and 4.2 we verify

$$
\begin{aligned}
\int N[P_a, \bar{g}^*(B), y]dy &= \int\int N\{h_u, [\alpha(u), \beta(u)]^0, v\}dvdu \\
&= \int\int N\{h_u, [\alpha(u), \beta(u)], v\}dvdu = \int T_{\alpha(u)}^{\beta(u)} h_u du \\
&= \int T_{v=\alpha(u)}^{\beta(u)}[a_3 v - a_2 g(u, v)]du.
\end{aligned}
$$

Part 3. $\Phi[\bar{g}^*(B)] \geqq \int T_{v=\alpha(u)}^{\beta(u)}[a_2 g(u, v) - a_3 v]du$.

From 4.4 we see that $\Phi[\bar{g}^*(B)] \geqq \int N[P_a, \bar{g}^*(B), y]dy$ and use Part 2 to complete the proof.

Part 4. $|P_a^*[\bar{g}^*(B)]| \leqq H(g, B)$.

Use 4.4, Part 2, 5.4 and 5.5 to check:

$$
\begin{aligned}
|P_a^*[\bar{g}^*(B)]| &\leqq \int N[P_a, \bar{g}^*(B), y]dy \\
&= \int T_{v=\alpha(u)}^{\beta(u)}[a_2 g(u, v) - a_3 v]du \\
&\leqq a_2 \int T_{v=\alpha(u)}^{\beta(u)} g(u, v)du + a_3 \int T_{v=\alpha(u)}^{\beta(u)} v du \\
&= a_2 H_1(g, B) + a_3 |B| \\
&\leqq \{a_2^2 + a_3^2\}^{1/2} \cdot \{[H_1(g, B)]^2 + |B|^2\}^{1/2} \\
&= \{[H_1(g, B)]^2 + |B|^2\}^{1/2} \leqq H(g, B).
\end{aligned}
$$

Part 5. If u is a number and $\{P_a^*(S)\}^u \neq 0$, then $B^u \neq 0$ and $|\{P_a^*(S)\}^u| \leqq |\{P_a^*[\bar{g}^*(B)]\}^u| + 2\delta$.

With the help of Part 1 we notice that

$$\begin{aligned}\{P_a^*[\bar{g}^*(B)]\}^u &= \underset{v}{E}\,[(u, v) \in P_a^*[\bar{g}^*(B)]]\\ &= \underset{v}{E}\,[N\{P_a, \bar{g}^*(B), (u, v)\} > 0]\\ &= \underset{v}{E}\,[N\{h_u, [\alpha(u), \beta(u)]^0, v\} > 0]\\ &= \underset{v}{E}\,[v \in h_u^*\{[\alpha(u), \beta(u)]^0\}]\end{aligned}$$

and infer from the continuity of h_u that

(3) $\{P_a^*[\bar{g}^*(B)]\}^u$ is an interval.

Now fix for a moment a point $v \in \{P_a^*(S)\}^u$. Pick z so that $z \in S$ and $P_a(z) = (u, v)$. From the definition of S and the relation (1) we infer

(4) $(z_1, z_2) \in B, \qquad |z_3 - g(z_1, z_2)| < \delta, \qquad z_1 = u, \qquad a_3z_2 - a_2z_3 = v,$

which implies $z_2 \in B^u$ and $B^u \neq 0$. Next use (2) and (4) to check that $P_a[\bar{g}(z_1, z_2)] = (z_1, a_3z_2 - a_2g(z_1, z_2)) = (u, a_3z_2 - a_2g(z_1, z_2))$, hence $a_3z_2 - a_2g(z_1, z_2) \in \{P_a^*[\bar{g}^*(B)]\}^u$. Apply (4) again to infer $|a_3z_2 - a_2g(z_1, z_2) - v| = |a_2z_3 - a_2g(z_1, z_2)| \leq |z_3 - g(z_1, z_2)| < \delta$. We have thus exhibited a point of $\{P_a^*[\bar{g}^*(B)]\}^u$ within a distance less than δ from v.

Combining (3) with the arbitrary nature of v, we easily see that the set $\{P_a^*(S)\}^u$ is contained in an interval of length $|\{P_a^*[\bar{g}^*(B)]\}^u| + 2\delta$.

Part 6. $|P_a^*(S)| \leq H(g, B) + 2\delta$ diam B.

Let

$$Q = \underset{u}{E}\,[\{P_a^*(S)\}^u \neq 0]$$

and use Part 5 to infer

$$Q \subset \underset{u}{E}\,[B^u \neq 0].$$

Accordingly

$$|Q| \leq \operatorname{diam} Q \leq \operatorname{diam} \underset{u}{E}\,[B^u \neq 0] \leq \operatorname{diam} B$$

and we combine Parts 4 and 5 with the last relation to conclude

$$\begin{aligned}|P_a^*(S)| &= \int_Q |\{P_a^*(S)\}^u|\,du \leq \int_Q (|\{P_a^*[\bar{g}^*(B)]\}^u| + 2\delta)du\\ &\leq \int |\{P_a^*[\bar{g}^*(B)]\}^u|\,du + 2\delta|Q|\\ &\leq |P_a^*[\bar{g}^*(B)]| + 2\delta \operatorname{diam} B \leq H(g, B) + 2\delta \operatorname{diam} B.\end{aligned}$$

6.3 LEMMA. *If g is continuous on E_2 to E_1, n is a positive integer, $\alpha<\beta$ and λ, μ, u are any three numbers, then* $T_{v=\alpha}^{\beta}[\lambda g_n(u, v)+\mu v] \leqq n^2\pi^{-1}\int_{K_n} T_{v=\alpha}^{\beta}[\lambda g(u+z_1, v+z_2)+\mu(v+z_2)]dz$.

Proof. If $\alpha=v_0\leqq v_1\leqq v_2\leqq \cdots \leqq v_m=\beta$, then

$$\sum_{i=1}^{m} |\lambda g_n(u, v_i) + \mu v_i - \lambda g_n(u, v_{i-1}) - \mu v_{i-1}|$$

$$= \sum_{i=1}^{m} \Big| n^2\pi^{-1} \int_{K_n} \{\lambda g(u + z_1, v_i + z_2) + \mu(v_i + z_2) - \lambda g(u + z_1, v_{i-1} + z_2) - \mu(v_{i-1} + z_2)\} dz \Big|$$

$$\leqq n^2\pi^{-1} \int_{K_n} \sum_{i=1}^{m} |\lambda g(u + z_1, v_i + z_2) + \mu(v_i + z_2) - \lambda g(u + z_1, v_{i-1} + z_2) - \mu(v_{i-1} + z_2)| \, dz$$

$$\leqq n^2\pi^{-1} \int_{K_n} T_{v=\alpha}^{\beta}[\lambda g(u + z_1, v + z_2) + \mu(v + z_2)]dz.$$

6.4 LEMMA. *If B is a bounded open convex subset of E_2, g is continuous on E_2 to E_1, n is a positive integer, $a\in E_3$, $|a|=1$, $a_1=0$, $a_2\geqq 0$, $a_3\geqq 0$, then* $n^2\pi^{-1}\int_{K_n}\Phi[\bar{g}^*(B_z)]dz \geqq \int_B |a_2D_2g_n(x)-a_3|\,dx$.

Proof. Let α and β be such functions that

$$B = \underset{(u,v)}{E}\,[\alpha(u) < v < \beta(u)]$$

and note that

$$B_z = \underset{(u,v)}{E}\,[\alpha(u - z_1) < v - z_2 < \beta(u - z_1)] \quad \text{for} \quad z \in E_2.$$

Next we use 6.3 and 6.2 to check:

$$\int_B |a_2D_2g_n(x) - a_3| \, dx = \int T_{v=\alpha(u)}^{\beta(u)}[a_2g_n(u, v) - a_3v]du$$

$$\leqq \int n^2\pi^{-1} \int_{K_n} T_{v=\alpha(u)}^{\beta(u)}[a_2g(u + z_1, v + z_2) - a_3(v + z_2)]dzdu$$

$$= n^2\pi^{-1} \int_{K_n} \int T_{v=\alpha(u)}^{\beta(u)}[a_2g(u + z_1, v + z_2) - a_3(v + z_2)]dudz$$

$$= n^2\pi^{-1} \int_{K_n} \int T_{v=\alpha(u-z_1)}^{\beta(u-z_1)}[a_2g(u, v + z_2) - a_3(v + z_2)]dudz$$

$$= n^2\pi^{-1} \int_{K_n} \int T_{v=\alpha(u-z_1)+z_2}^{\beta(u-z_1)+z_2}[a_2g(u, v) - a_3v]dudz$$

$$\leqq n^2\pi^{-1} \int_{K_n} \Phi[\bar{g}^*(B_z)]dz.$$

6.5 LEMMA. *If A is a bounded open convex subset of E_2, f is continuous on E_2 to E_1, $\delta>0$,*

$$R = E_3 \mathop{E}_{z} [(z_1, z_2) \in A, \ |z_3 - f(z_1, z_2)| < \delta];$$

n is a positive integer; $c \in E_3$, $|c| = 1$, $c_3 \geqq 0$, then

$$|P_c^*(R)| \leqq H(f, A) + 2\delta \operatorname{diam} A,$$

$$n^2\pi^{-1} \int_{K_n} \Phi[\bar{f}^*(A_z)]dz \geqq \int_A |c_1D_1f_n(x) + c_2D_2f_n(x) - c_3| \, dx.$$

Proof. If $c_1=c_2=0$, the lemma follows immediately from 6.2 and 6.4. Hence we assume $c_1^2+c_2^2>0$ and select such points a and b that:

$$c = (-a_2b_2, a_2b_1, a_3);$$
$$a \in E_3, \ |a| = 1, \ a_1 = 0, \ a_2 \geqq 0, \ a_3 \geqq 0; b \in E_2, \ |b| = 1.$$

Let T be the rotation such that

$$T(x) = (b_1x_1 - b_2x_2, b_2x_1 + b_1x_2) \quad \text{for} \quad x \in E_2$$

and denote the inverse of T by U. We further define

$$g = f:T, \qquad B = U^*(A),$$
$$S = E_3 \mathop{E}_{z} [(z_1, z_2) \in B, \ |z_3 - g(z_1, z_2)| < \delta]$$

and divide the remainder of the proof into five parts.

Part 1. $P_c^*(R) \subset P_a^*(S)$.

Let $x \in P_c^*(R)$. There is a point z for which

$$(z_1, z_2) \in A, \qquad |z_3 - f(z_1, z_2)| < \delta, \qquad P_c(z) = x.$$

Define $w \in E_3$ by the relations

$$(w_1, w_2) = U(z_1, z_2), \qquad w_3 = z_3$$

and note that

$$(w_1, w_2) \in B, \qquad |w_3 - [f:T](w_1, w_2)| < \delta,$$

which implies $w \in S$. Now

$$\begin{aligned} x = P_c(z) &= \{c_1^2 + c_2^2\}^{1/2} (c_2z_1 - c_1z_2, c_3(c_1z_1 + c_2z_2) - (c_1^2 + c_2^2)z_3) \\ &= a_2^{-1}(a_2(b_1z_1 + b_2z_2), a_3a_2(-b_2z_1 + b_1z_2) - a_2^2z_3) \\ &= (w_1, a_3w_2 - a_2w_3) = P_a(w) \end{aligned}$$

so that $x \in P_a^*(S)$.

Part 2. $|P_a^*(R)| \leqq H(f, A) + 2\delta \operatorname{diam} A$.

We use Part 1, 6.2 and 5.6 in checking that

$$\begin{aligned} | P_c^*(R) | \leqq | P_a^*(S) | &\leqq H(g, B) + 2\delta \operatorname{diam} B \\ &= H(f:T, B) + 2\delta \operatorname{diam} B \\ &= H[f, T^*(B)] + 2\delta \operatorname{diam} U^*(A) = H(f, A) + 2\delta \operatorname{diam} A. \end{aligned}$$

Part 3. $D_2 g_n(x) = -b_2 D_1 f_n[T(x)] + b_1 D_2 f_n[T(x)]$.

$$\begin{aligned} g_n(x) &= n^2\pi^{-1}\int_{K_n} g(x+z)dz = n^2\pi^{-1}\int_{K_n} f[T(x+z)]dz \\ &= n^2\pi^{-1}\int_{K_n} f[T(x)+T(z)]dz = n^2\pi^{-1}\int_{K_n} f[T(x)+z]dz \\ &= f_n[T(x)] = f_n(b_1x_1 - b_2x_2,\ b_2x_1 + b_1x_2) \end{aligned}$$

and the conclusion is now evident.

Part 4. $A_z = T^*[B_{U(z)}]$.

$$\begin{aligned} T^*[B_{U(z)}] &= T^*\{\underset{x}{E}[x - U(z) \in B]\} \\ &= \underset{y}{E}[U(y) - U(z) \in B] = \underset{y}{E}[y - z \in T^*(B)] = A_z. \end{aligned}$$

Part 5. $n^2\pi^{-1}\int_{K_n}\Phi[\bar{f}^*(A_z)]dz \geqq \int_A |c_1D_1f_n(x) + c_2D_2f_n(x) - c_3|dx$.

From Part 4, 6.4 and Part 3 it follows that

$$\begin{aligned} n^2\pi^{-1}\int_{K_n}\Phi[\bar{f}^*(A_z)]dz &= n^2\pi^{-1}\int_{K_n}\Phi\{\bar{f}^*[T^*(B_{U(z)})]\}dz \\ &= n^2\pi^{-1}\int_{K_n}\Phi\{\bar{g}^*[B_{U(z)}]\}dz = n^2\pi^{-1}\int_{K_n}\Phi[\bar{g}^*(B_z)]dz \\ &\geqq \int_B |a_2D_2g_n(x) - a_3|dx \\ &= \int_B |-a_2b_2D_1f_n[T(x)] + a_2b_1D_2f_n[T(x)] - a_3|dx \\ &= \int_{T^*(B)} |-a_2b_2D_1f_n(x) + a_2b_1D_2f_n(x) - a_3|dx \\ &= \int_A |c_1D_1f_n(x) + c_2D_2f_n(x) - c_3|dx. \end{aligned}$$

6.6 LEMMA. *If f is continuous on E_2 to E_1, R is an open rectangle and n is a positive integer, then*

$$n^2\pi^{-1}\int_{K_n}\Phi[\bar{f}^*(R_z)]dz \geqq H(f_n, R).$$

Proof. Let $\epsilon>0$. Denote

$$p(x, y) = \frac{|D_1f_n(x)D_1f_n(y) + D_2f_n(x)D_2f_n(y) + 1|}{([D_1f_n(y)]^2 + [D_2f_n(y)]^2 + 1)^{1/2}}$$

and choose $\delta>0$ so that

$$|x - y| < \delta \quad \text{implies} \quad |p(x, y) - p(x, x)| < \epsilon$$

whenever x and y are points of R.

Select disjoint open rectangles $A^1, A^2, \cdots, A^m$, each of diameter less than δ, such that

$$\sum_{i=1}^{m} A^i \subset R, \qquad \sum_{i=1}^{m} H(f_n, A^i) = H(f_n, R).$$

For $i=1, 2, \cdots, m$ choose points $y^i \in A^i$ and take $c^i \in E_3$ so that

$$c^i = \frac{(-D_1f_n(y^i), -D_2f_n(y^i), 1)}{([D_1f_n(y^i)]^2 + [D_2f_n(y^i)]^2 + 1)^{1/2}}.$$

Now 6.5 helps us to conclude

$$\begin{aligned}
n^2\pi^{-1}\int_{K_n}\Phi[\bar{f}^*(R_z)]dz &\geqq n^2\pi^{-1}\int_{K_n}\Phi\left[\bar{f}^*\left(\sum_{i=1}^{m}A^i_z\right)\right]dz \\
&= n^2\pi^{-1}\int_{K_n}\Phi\left[\sum_{i=1}^{m}\bar{f}^*(A^i_z)\right]dz \\
&= n^2\pi^{-1}\int_{K_n}\sum_{i=1}^{m}\Phi[\bar{f}^*(A^i_z)]dz \\
&= \sum_{i=1}^{m} n^2\pi^{-1}\int_{K_n}\Phi[\bar{f}^*(A^i_z)]dz \\
&\geqq \sum_{i=1}^{m}\int_{A^i}|c^i_1D_1f_n(x) + c^i_2D_2f_n(x) - c^i_3|\,dx \\
&= \sum_{i=1}^{m}\int_{A_i}p(x, y^i)dx > \sum_{i=1}^{m}\int_{A^i}[p(x, x) - \epsilon]dx \\
&= \sum_{i=1}^{m}H(f_n, A^i) - \epsilon\sum_{i=1}^{m}|A^i| \geqq H(f_n, R) - \epsilon|R|.
\end{aligned}$$

Since ϵ was an arbitrary positive number the proof is complete.

6.7 LEMMA. *If f is continuous on E_2 to E_1, $S\subset R$, and S and R are open rectangles with disjoint boundaries, then*

$$\Phi[\bar{f}^*(R)] \geqq H(f, S).$$

Proof. Choose ν so large that $n>\nu$ implies

$$S_z \subset R \quad \text{for} \quad z \in K_n.$$

Hence we may use 6.6 to infer

$$\Phi[\bar{f}^*(R)] \geqq n^2\pi^{-1}\int_{K_n} \Phi[\bar{f}(S_z)]dz \geqq H(f_n, S)$$

for $n>\nu$. From this and the lower-semicontinuity of H we conclude

$$\Phi[\bar{f}^*(R)] \geqq \liminf_{n\to\infty} H(f_n, S) \geqq H(f, S).$$

6.8 Lemma. *If f is continuous on E_2 to E_1 and R is an open rectangle, then*

$$\Phi[\bar{f}^*(R)] \leqq H(f, R).$$

Proof. We may assume $H(f, R) < \infty$.

Let $r>0$. Choose a number ρ so that $0<\rho<r/4$ and $|f(x)-f(z)|<r/4$ whenever $x\in R$, $z\in R$, $|z-x|<\rho$.

Select open rectangles $A^1, A^2, \cdots, A^m$, each with diameter less than δ, such that

$$R \subset \sum_{i=1}^{m} A^i, \qquad \sum_{i=1}^{m} H(f, A^i) \leqq H(f, R) + r.$$

This can be done in virtue of the continuity of H.

Further choose such a number δ that

$$0 < \delta < \rho, \qquad \sum_{i=1}^{m} 2\delta \cdot \operatorname{diam} A^i < r$$

and define

$$B^i = E_3 \mathop{E}_{z} [(z_1, z_2) \in A^i, \ |z_3 - f(z_1, z_2)| < \delta] \quad \text{for} \quad i = 1, 2, \cdots, m.$$

Note that $B^1, B^2, \cdots, B^m$ are open connected sets, each with diameter less than r, and that

$$\bar{f}^*(R) \subset \sum_{i=1}^{m} B^i.$$

From 6.5 we further deduce that

$$\gamma(B^i) = \sup_{c\in E_3, |c|=1, c_3 \geqq 0} |P_c^*(B^i)| \leqq H(f, A^i) + 2\delta \operatorname{diam} A^i$$

for $i=1, 2, \cdots, m$. Hence

$$\gamma_r[\bar{f}^*(R)] \leqq \sum_{i=1}^{m} \gamma(B^i) \leqq \sum_{i=1}^{m} H(f, A^i) + \sum_{i=1}^{m} 2\delta \operatorname{diam} A^i \leqq H(f, R) + 2r.$$

Accordingly

$$\gamma_r[\bar{f}^*(R)] \leqq H(f, R) + 2r \quad \text{for} \quad r > 0.$$

Let $r \to 0$.

6.9 THEOREM. *If f is continuous on E_2 to E_1 and R is an open rectangle, then*

$$\Phi[\bar{f}^*(R)] = H(f, R).$$

Proof. Let S be such an ascending sequence of open rectangles that

$$R = \sum_{n=1}^{\infty} S_n$$

and for each positive integer n the boundaries of S_n and R are disjoint. Use 5.2, 6.7, 6.8 to conclude

$$H(f, R) = \lim_{n\to\infty} H(f, S_n) \leqq \Phi[\bar{f}^*(R)] \leqq H(f, R).$$

6.10 *Remark.* Combining 6.9 with Radó's theorem referred to in 5.6, we arrive at the following conclusion:

If f is continuous on E_2 to E_1 and R is an open rectangle, then $\Phi[\bar{f}^*(R)]$ equals the Lebesgue area of the surface defined by f above R.

Accordingly Tonelli's theorem (Saks, page 182) applies not only to Lebesgue area but also to our surface measure.

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